Goals of the course: We will build up the mathematical "technology" to generalize the Fundamental theoeem of Calulus (FTOC)

Recall:

If f is continuous & F is an antiderivative of f then $\int_a^b f(t)\,dt = F(b) - F(a)$

It turns out that there are important generalizations of the FTOC to the case of functions of several variables.

These generalizations have very important applications in engineering, physics, ..

For example, Maxwell's equations rely on vector calculus very heavily.

The plan (for the first part of the course)

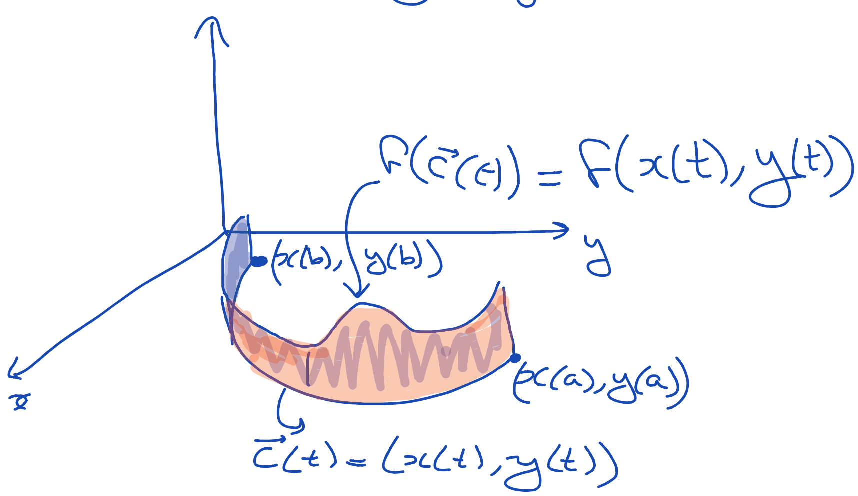
- Review
  - × Differentiation
  - × Double Integrals
  - × Triple Integrals

- Linear Maps & the change of variable formula
  - × Cylindrical & Spherical Coordinates

- Vector Fields

- Integrals over paths & Surfaces

Example (2D) Area of a fence. Let $f:\mathbb{R}^2 \to \mathbb{R}$



We'll learn how to compute this.

• Fund. Theorem of Line Integrals

$$\int_{\vec{c}} (\nabla f) \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$$

Let's get started

Review:

2.3 Differentiation

Recall: If $f: \mathbb{R} \to \mathbb{R}$

$$\frac{df}{dx}(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Recall (partial derivatives)

If $f: \mathbb{R}^3 \to \mathbb{R}$

then $\displaystyle \frac{\partial f}{\partial x}(x,y,z) = \lim_{h \to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$

$$\frac{\partial f}{\partial y}(x,y,z) = \lim_{h \to 0} \frac{f(x,y+h,z) - f(x,y,z)}{h}$$

(What is $\frac{\partial f}{\partial z}$?)

Notice that the partial derivatives are themselves functions of x, y, z

e.g: $f(x,y,z) = x^2y^3 + \sin(xy) + z^2y$

find $\frac{\partial f}{\partial y}(0,1,1)$

soln: $\frac{\partial f}{\partial y} = \frac{\partial(x^2y^3 + \sin xy + z^2y)}{\partial y} = 3x^2y^2 + x\cos xy + z^2$

$\Rightarrow \frac{\partial f}{\partial y}(0,1,1) = 3(0)(1) + 0\cos 0 + 1 = 1$

— x —

Recall: Def'n of differentiability

Let's start with 2 variables:

Def'n: Let $f: \mathbb{R}^2 \to \mathbb{R}$. We say that f is differentiable at (x_0, y_0) if $\frac{\partial f}{\partial x}(x_0,y_0)$ and $\frac{\partial f}{\partial y}(x_0,y_0)$ exist and if

$$\frac{f(x,y) - f(x_0,y_0) - \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) - \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0)}{\|(x,y)-(x_0,y_0)\|} \longrightarrow 0$$

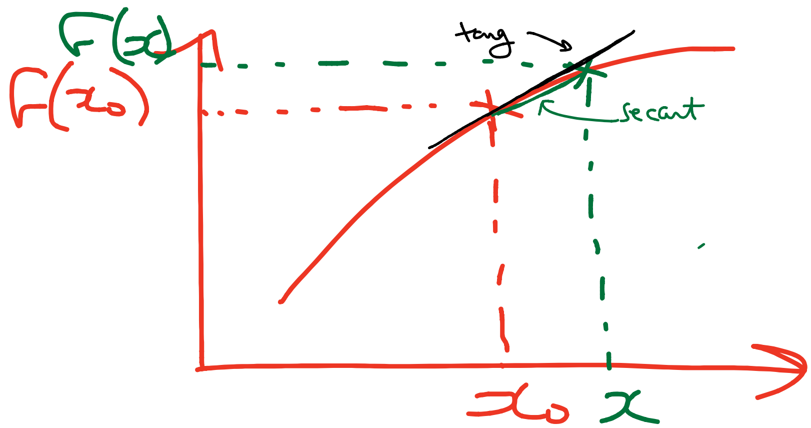
as $(x,y) \longrightarrow (x_0,y_0)$

But what does this mean?

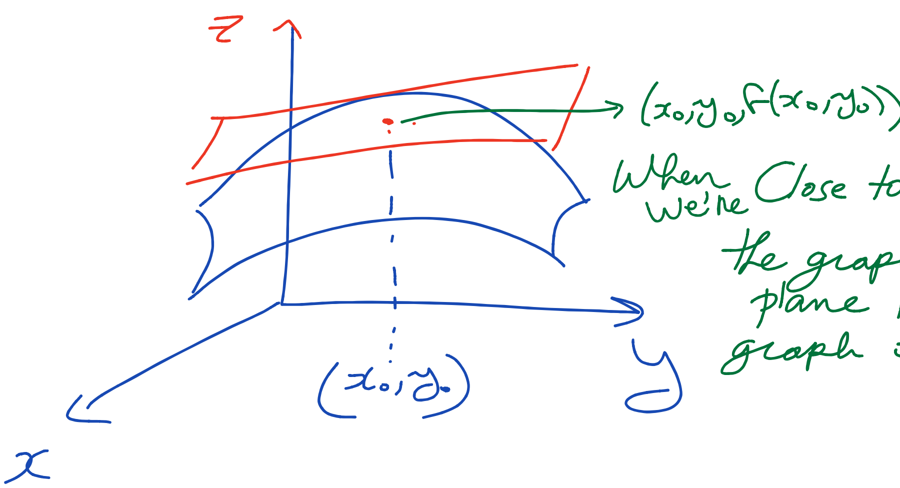
<u>one variable</u>: $\dfrac{f(x) - f(x_0) - \left[\frac{\partial f}{\partial x}(x_0)\right](x - x_0)}{x - x_0} \longrightarrow 0 \quad \text{as } x \to x_0$

is equivalent to $\underbrace{\dfrac{f(x) - f(x_0)}{x - x_0}}_{\text{slope of secant}} \longrightarrow \underbrace{\dfrac{\partial f}{\partial x}(x_0)}_{\text{slope of tangent}} \quad \text{as } x \to x_0$

and to $f(x) \longrightarrow \underbrace{\left[\frac{\partial f}{\partial x}(x_0)\right](x - x_0) + x_0}_{\substack{\text{linear approx.} \\ \text{at } x_0}} \quad \text{as } x \to x_0$



Two variables



When we're close to (x_0, y_0) the graph of the tangent plane is close to the graph of f.

Two variables: $z = f(x_0,y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](x-x_0)$

the linear approximation of f at (x_0,y_0): (This is a plane)

is a good app. of $f(x)$ when $(x,y) \to (x_0,y_0)$

Definition: (tangent plane) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be differentiable at (x_0,y_0)

The plane in $\mathbb{R}^3$ given by

$$z = f(x_0,y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0)$$

is called the tangent plane to the graph of f at $(x_0,y_0,f(x_0,y_0))$

Example: Find the plane tangent to the graph of $f(x,y) = x + y^2 + \cos(xy)$ at $(0,1)$

Sol'n: The eq'n of the tg. plane is given by

$$z = f(0,1) + \frac{\partial f}{\partial x}(0,1)\,(x-0) + \frac{\partial f}{\partial y}(0,1)\,(y-1)$$

need these

- $f(0,1) = 0 + 1 + \cos(0) = 2$

- $\frac{\partial f}{\partial x} = 1 + 0 - y\sin(xy) \Rightarrow \frac{\partial f}{\partial x}(0,1) = 1 - 1\sin(0) = 1$

$\bullet\ \frac{\partial f}{\partial y} = 2y - x\sin(xy) \Rightarrow \frac{\partial f}{\partial y}(0,1) = 2(1) - 0\sin(0\times 1)$

$= 2$

So the eq'n of the plane is

$$z = 2 + 1(x-0) + 2(y-1) = x + 2y$$

—— x ——

Differentiability: The general case

The derivative of $f: \mathbb{R}^n \to \mathbb{R}^m$ at $\vec{x}_0$ denoted $Df(\vec{x}_0)$

is a matrix T whose elements are $t_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\vec{x}_0}$

So if $f = (f_1, f_2, \cdots, f_m)$

$$\underbrace{T = \underbrace{Df(\vec{x}_0)}}_{\text{derivative or differential of } f \text{ at } \vec{x}_0} = \underbrace{\begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{\vec{x}_0} & \left.\frac{\partial f_2}{\partial x_2}\right|_{\vec{x}_0} & \cdots & \left.\frac{\partial f_1}{\partial x_n}\right|_{\vec{x}_0} \\ \vdots & & & \\ \left.\frac{\partial f_m}{\partial x_1}\right|_{\vec{x}_0} & \left.\frac{\partial f_m}{\partial x_2}\right|_{\vec{x}_0} & \cdots & \left.\frac{\partial f_m}{\partial x_n}\right|_{\vec{x}_0} \end{bmatrix}}_{\text{matrix of partial derivatives}}$$

Definition (General case)

Let U be an open set in $\mathbb{R}^n$ & let $f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a given ft.

f is differentiable at $\vec{x}_0 \in U$ if the partial derivatives of f exist at $\vec{x}_0$ and if:

$$\lim_{\vec{x}\to\vec{x}_0} \frac{\| f(\vec{x}) - f(\vec{x}_0) - \underbrace{T}_{m\times n \text{ matrix}} \overbrace{(\vec{x}-\vec{x}_0)}^{n\times 1 \text{ vector}} \|}{\| x - x_0 \|} = 0$$

Example: $f(x,y,z) = (x^2+\cos y, -ye^z)$

Find $Df(x,y,z)$

$$Df(x,y,z) = \begin{bmatrix} \frac{\partial(x^2+\cos y)}{\partial x} & \frac{\partial(x^2+\cos y)}{\partial y} & \frac{\partial(x^2+\cos y)}{\partial z} \\ \frac{\partial(-ye^z)}{\partial x} & \frac{\partial(-ye^z)}{\partial y} & \frac{\partial(-ye^z)}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} 2x & -\sin y & 0 \\ 0 & -e^z & -ye^z \end{bmatrix}$$

Remark: If $f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ then $Df(x_0)$ is a $1\times n$ matrix.

The corresponding vector $\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$ is called the gradient and denoted ∇f.

e.g. $f(x,y,z) = 2x + y^2 + ze^x$

$$\Rightarrow \nabla f = (2+e^x)\vec{i} + (2y)\vec{j} + (e^x)\vec{k}.$$

Theorem: If $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{x}_0 \in U$, then f is cont. at $\vec{x}_0$.

Theorem: If $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ is such that

- the partials $\frac{\partial f_i}{\partial x_j}$ all exist
- and are continuous in a neighborhood of $\vec{x} \in U$

then f is differentiable at $\vec{x}$.

much easier to check than the definition of differentiability.

Example: $f(x,y,z) = (x^2 + \cos y, -ye^z)$

from the prev. example is differentiable

because the partials exist & are cont.

2.5 Properties of the derivative

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ & $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\vec{x}_0$. Then:

(i) Constant Multiple rule: if $c \in \mathbb{R}$ and $h(\vec{x}) = c f(\vec{x})$, then $h(\vec{x})$ is differentiable at $\vec{x}_0$ &

$$Dh(\vec{x}_0) = c\, Df(\vec{x}_0)$$

(ii) Sum rule: if $h(\vec{x}) = f(\vec{x}) + g(\vec{x})$ then h is differentiable at $\vec{x}_0$

and $Dh(\vec{x}_0) = \underbrace{Df(\vec{x}_0)}_{1\times n \text{ matrix}} + \underbrace{Dg(\vec{x}_0)}_{1\times n \text{ matrix}}$

(iii) Product Rule.

If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ & $g : U \subset \mathbb{R}^n \to \mathbb{R}$ are differentiable at $\vec{x}_0$ and $h(\vec{x}) = f(\vec{x}) g(\vec{x})$

then h is differentiable at $\vec{x}_0$ &

$$Dh(\vec{x}_0) = \underbrace{g(x_0)}_{\in \mathbb{R}} \underbrace{Df(x_0)}_{1\times n \text{ matrix}} + \underbrace{f(x_0)}_{\in \mathbb{R}} \underbrace{Dg(x_0)}_{1\times n \text{ matrix}}$$

e.g. Let $f(x,y) = x^2 y$ and $g(x,y) = x\sin y$.
Let $h(x,y) = f(x,y)g(x,y)$.
Find $(Dh)(1,0)$.

Sol'n

$$Dh(x,y) = g(x,y)(Df)(x,y) + f(x,y)(Dg)(x,y)$$

$$= g(x,y)\left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}\right] + f(x,y)\left[\frac{\partial g}{\partial x} \quad \frac{\partial g}{\partial y}\right]$$

$$= x\sin y\left[2xy \quad x^2\right] + x^2 y\left[\sin y \quad x\cos y\right]$$

$$\Rightarrow Dh(1,0) = (1)(0)[0 \quad 1] + 0[0 \quad 1] = 0$$

(iv) Quotient Rule

suppose further that $g(\vec{x}) \neq 0 \quad \forall \vec{x} \in U$

if $h(x) = f(x)/g(x)$ then h is differentiable at $\vec{x}_0$ &

$$Dh(\vec{x}_0) = \frac{g(\vec{x}_0)Df(\vec{x}_0) - f(\vec{x}_0)Dg(\vec{x}_0)}{[g(\vec{x}_0)]^2}$$

(V) Chain Rule

Let $f: V \subset \mathbb{R}^m \to \mathbb{R}^p$ & (V ↑ open set)

$g: U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$. ($U$ ↑ open set) Let g be differentiable at $\vec{x}_0$

and f be diff at $\vec{y}_0 = g(\vec{x}_0)$. Then

$$D(f\circ g)(x_0) = [\underbrace{Df(y_0)}_{p\times m}][\underbrace{Dg(x_0)}_{m\times n}] \quad (p\times n)$$

makes sense bec $h = f\circ g: \mathbb{R}^n \to \mathbb{R}^p$

Compare to single variable calculus $\dfrac{df(g(x))}{dx} = f'(\underbrace{g(x)}_{y})\, g'(x)$

Example: $g(x,y) = (x^2+1, y^2)$ & $f(u,v) = (u+v, u, v^2)$

compute the derivative of $f\circ g$ at $(1,1)$ using the chain rule

$Dg(x,y) = \begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix}$, $Df(u,v) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2v \end{bmatrix}$, $g(1,1) = (2,1)$

$$D(f\circ g)(1,1) = Df(g(1,1)) \cdot Dg(1,1)$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

Special Case of the chain rule

$c : \mathbb{R} \to \mathbb{R}^3$ is a diff. path, $f : \mathbb{R}^3 \to \mathbb{R}$, $h(t) = f(c(t))$

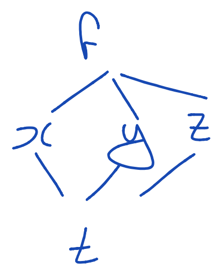
$c(t) = (x(t), y(t), z(t))$ $\qquad = f(x(t), y(t), z(t))$

then $\dfrac{dh}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\cdot\dfrac{dy}{dt} + \dfrac{\partial f}{\partial z}\cdot\dfrac{dz}{dt} = \nabla f(\vec{c(t)}) \cdot \vec{c'(t)}$

$$= \underbrace{[D f(c(t))]}_{1\times 3}\underbrace{[D c(t)]}_{3\times 1}$$

"The trick"

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

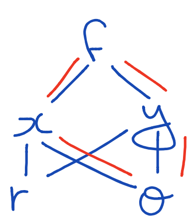


example: Let $f(x,y)$ be given, subst. $x = r\cos\theta$, $y = r\sin\theta$

find $\dfrac{\partial f}{\partial \theta}$ & $\dfrac{\partial f}{\partial r}$.

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta} = (-r\sin\theta)\frac{\partial f}{\partial x} + (r\cos\theta)\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}$$



3.2 Taylor's Theorem

We did ~~not~~ cover this in class, but it is on the HW

Single variable: $f(x_0+h) = \underbrace{\overbrace{f(x_0) + f'(x_0)h}^{\text{linear approx.}} + f''(x_0)\,h^2/2}_{\text{quadratic approx.}} + \cdots + \frac{f^{(k)}(x_0)}{k!}h^k$

$+ \underbrace{R_k(x_0,h)}_{\lim_{h\to 0}\frac{R_k(x_0,h)}{h^k}=0} \leftarrow$ remainder

multi-variable:

$f: \mathbb{R}^n \to \mathbb{R}$

linear approx (we saw this before): $(k=1)$

$$f(\vec{x}_0+\vec{h}) = f(\vec{x}_0) + \nabla f(\vec{x}_0)\cdot\vec{h} + R_1(\vec{x}_0,\vec{h})$$

$$= f(\vec{x}_0) + \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\vec{x}_0)\cdot h_i + \underbrace{R_1(\vec{x}_0,\vec{h})}_{\lim_{h\to 0}\frac{R_1(\vec{x}_0,h)}{\|h\|}=0}$$

quadratic approx: $(k=2)$

$$f(\vec{x}_0+\vec{h}) = f(\vec{x}_0) + \nabla f(\vec{x}_0)\cdot\vec{h} + \frac{1}{2}(h_1,h_2,\dots,h_n)\underbrace{\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\vec{x}) & \frac{\partial^2 f}{\partial x_1\partial x_2}(\vec{x}) & \cdots & \frac{\partial^2 f}{\partial x_1\partial x_n}(\vec{x}) \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1^2}(\vec{x}) & \frac{\partial^2 f}{\partial x_1\partial x_2}(\vec{x}) & \cdots & \frac{\partial^2 f}{\partial x_1\partial x_n}(\vec{x}) \end{bmatrix}}_{\substack{H:\ \text{Hessian} \\ (\text{matrix of second derivatives}) \\ n\times n}}\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$$

$$+ \underbrace{R_2(\vec{x}_0,\vec{h})}_{\lim_{h\to 0}\frac{R_2(\vec{x}_0,\vec{h})}{\|h\|_2}=0}$$

Alternatively: $f(\vec{x}_0+\vec{h}) = f(\vec{x}_0) + \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\vec{x}_0)\,h_i + \frac{1}{2}\underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n}}_{n^2\text{ terms}}\frac{\partial^2 f}{\partial x_i\partial x_j}(\vec{x}_0)\,h_i h_j$

$+ R_2(\vec{x}_0,\vec{h})$

example: $f(x,y) = \sin(x+2y)$. Find the second order taylor formula about $x_0 = (0,0)$.

We need: $f(0,0)$, $\frac{\partial f}{\partial x}(0,0)$, $\frac{\partial f}{\partial y}(0,0)$, $\frac{\partial^2 f}{\partial x^2}(0,0)$, $\frac{\partial^2 f}{\partial y^2}(0,0)$, $\frac{\partial^2 f}{\partial x \partial y}(0,0)$

- $f(0,0) = 0$
- $\frac{\partial f}{\partial x} = \cos(x+2y) \Rightarrow \partial f/\partial x(0,0) = 1$, $\frac{\partial f}{\partial y}(0,0) = 2\cos(0) = 2$
- $\frac{\partial^2 f}{\partial x^2} = -\sin(x+2y) \Rightarrow \frac{\partial^2 f}{\partial x^2}(0,0) = 0$, $\frac{\partial^2 f}{\partial y^2} = -4\sin(x+2y) \Rightarrow \frac{\partial^2 f}{\partial y^2}(0,0) = 0$
- $\frac{\partial^2 f}{\partial x \partial y} = -2\sin(x+2y) \Rightarrow \frac{\partial^2 f}{\partial x \partial y} = 0$

So, we have $H\Big|_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

So $f(h_1,h_2) = 0 + 1h_1 + 2h_2 + 0 + \underbrace{R_2(\vec{0},\vec{h})}_{\lim_{h\to 0} \frac{R_2(0,h)}{\|h\|^2} \to 0}$

example: Determine the second order taylor formular for $f(x,y) = e^{x+y}$ about $(0,0)$

solution: $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = e^{x+y}$ & $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x \partial y} = e^{x+y}$

$$\Rightarrow f(h_1,h_2) = f(0,0) + \nabla f(0,0)\cdot\vec{h} + \frac{1}{2}(h_1,h_2)\,H\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + R_2(\vec{0},\vec{h})$$

$$= 1 + (1,1)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \frac{1}{2}(h_1,h_2)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + R_2(\vec{0},\vec{h})$$

$$= 1 + h_1 + h_2 + \frac{1}{2}(h_1+h_2,\, h_2+h_2)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + R_2(\vec{0},\vec{h})$$

$$= 1 + h_1 + h_2 + \frac{1}{2}h_1^2 + \frac{1}{2}h_2^2 + h_1 h_2 + R_2(\vec{0},\vec{h})$$